Lagrangian Mechanics and Variational Calculus

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Abstract

This report reorganises a Year 1 presentation on Lagrangian mechanics and the calculus of variations into a written project note. The central aim is to explain why ordinary calculus is sufficient for finite-dimensional equilibrium problems, but variational calculus is needed when the unknown object is an entire function or path. The report begins with a spring-mass example, introduces functionals and the Euler–Lagrange equation, and then applies the method to the shortest path problem and the brachistochrone problem.

1 From Ordinary Calculus to Variational Problems

In ordinary calculus, extrema of a differentiable function $f(x)$ are found by solving

$$f'(x) = 0,$$

then testing the critical points to decide whether they are local minima, local maxima, or saddle points. This method applies when the variable is a real number or a finite list of real numbers.

Many physical questions have a different structure. Instead of asking for the best value of a single number, one asks for the best path, curve, or trajectory. In that setting the unknown object is a function, and the quantity to optimise is a functional: a map from functions to real numbers.

2 A Finite-Dimensional Example: Spring-Mass Equilibrium

Consider a mass m hanging from an ideal spring with spring constant k. Let x be the downward displacement from the spring's natural length. The forces are

$$F_{\text{gravity}} = mg, \qquad F_{\text{spring}} = -kx.$$

At static equilibrium, Newton's second law gives

$$mg - kx = 0, \qquad x = \frac{mg}{k}.$$

The same result can be obtained by minimising the potential energy

$$U(x) = \frac{1}{2}kx^2 - mgx.$$

Differentiating,

$$\frac{dU}{dx} = kx - mg.$$

Setting $dU/dx = 0$ gives $x = mg/k$. Since

$$\frac{d^2U}{dx^2} = k > 0,$$

this critical point is a stable minimum. Graphically, $U(x)$ is an upward-opening parabola with its minimum at $x = mg/k$.

This example is important because it shows the logic that will later reappear in a more general form: physical behaviour can often be characterised as the stationary point of an energy-like quantity.

3 Functionals

A functional maps a function to a real number. A common form is

$$J[y] = \int_{x_0}^{x_1} F(x, y(x), y'(x))\, dx,$$

where y is the unknown function and F is called the integrand or Lagrangian density for the variational problem.

The variational problem is to find a function y satisfying boundary conditions such as

$$y(x_0) = y_0, \qquad y(x_1) = y_1,$$

so that $J[y]$ is minimised, maximised, or made stationary.

4 Derivation of the Euler–Lagrange Equation

Let y be a candidate stationary path. Perturb it by a small variation

$$y_\varepsilon(x) = y(x) + \varepsilon\eta(x),$$

where $\eta(x)$ is an arbitrary smooth function satisfying the fixed-endpoint conditions

$$\eta(x_0) = \eta(x_1) = 0.$$

Define

$$\Phi(\varepsilon) = J[y_\varepsilon] = \int_{x_0}^{x_1} F(x, y + \varepsilon\eta, y' + \varepsilon\eta')\, dx.$$

If y is stationary, then

$$\Phi'(0) = 0.$$

Differentiating under the integral sign gives

$$\Phi'(0) = \int_{x_0}^{x_1} \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta' \right) dx.$$

Use integration by parts on the second term:

$$\int_{x_0}^{x_1} \frac{\partial F}{\partial y'}\eta'\, dx = \left[\frac{\partial F}{\partial y'}\eta \right]_{x_0}^{x_1} - \int_{x_0}^{x_1} \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) \eta\, dx.$$

The boundary term vanishes because $\eta(x_0) = \eta(x_1) = 0$. Therefore

$$\Phi'(0) = \int_{x_0}^{x_1} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) \right] \eta\, dx.$$

Since η is arbitrary, the bracketed term must vanish. Hence the Euler–Lagrange equation is

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) = 0.}$$

5 Example: The Shortest Path Between Two Points

Let $y(x)$ be a curve joining two fixed points (x_0, y_0) and (x_1, y_1). The arc length is

$$S[y] = \int_{x_0}^{x_1} \sqrt{1 + (y')^2}\, dx.$$

Here

$$F(y') = \sqrt{1 + (y')^2}.$$

Because F has no explicit dependence on y, the Euler–Lagrange equation becomes

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0.$$

Thus

$$\frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1 + (y')^2}} = C,$$

where C is constant. Therefore y' is constant, so

$$y(x) = ax + b.$$

The shortest curve between two points in the plane is a straight line.

6 The Beltrami Identity

When the integrand $F(x, y, y')$ has no explicit dependence on x, the Euler–Lagrange equation implies the Beltrami identity

$$\boxed{F - y'\frac{\partial F}{\partial y'} = C.}$$

To derive it, compute

$$\frac{dF}{dx} = \frac{\partial F}{\partial y} y' + \frac{\partial F}{\partial y'} y'',$$

because $\partial F/\partial x = 0$. From the Euler–Lagrange equation,

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right).$$

Substitute this into dF/dx:

$$\frac{dF}{dx} = y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + y''\frac{\partial F}{\partial y'} = \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right).$$

Therefore

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0,$$

which proves the identity.

7 Example: The Brachistochrone Problem

The brachistochrone problem asks for the curve along which a particle, moving under gravity and without friction, travels from one point to another in the shortest time.

Assume the particle starts from rest and let y be the vertical drop. Conservation of energy gives

$$\frac{1}{2}mv^2 = mgy, \qquad v = \sqrt{2gy}.$$

The time element is

$$dt = \frac{ds}{v} = \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}\,dx.$$

The total travel time is therefore

$$T[y] = \int_{x_0}^{x_1} \sqrt{\frac{1+(y')^2}{2gy}}\,dx.$$

The integrand has no explicit x-dependence, so the Beltrami identity applies. Solving the resulting differential equation gives a cycloid. A standard parametrisation is

$$x = a(\theta - \sin\theta), \qquad y = a(1-\cos\theta),$$

where a and the endpoint value of θ are determined by the boundary conditions.

This result is counterintuitive: the fastest path is not a straight line. It initially drops steeply to gain speed, then flattens as it approaches the endpoint.

8 Comparison

Feature	Ordinary calculus	Variational calculus
Unknown	A number or finite vector	A function or path
Object optimised	Function $f(x)$	Functional $J[y]$
Main condition	$f'(x) = 0$	Euler–Lagrange equation
Example	Spring-mass equilibrium	Shortest path, brachistochrone
Typical output	Critical point	Differential equation for a path

9 Conclusion

Ordinary calculus and variational calculus share the same conceptual goal: identify stationary points. The difference lies in the object being varied. For finite-dimensional problems, the derivative of a function is enough. For path-dependent physical problems, the unknown is a whole function, and the Euler–Lagrange equation provides the corresponding stationarity condition.

This transition is one of the main reasons Lagrangian mechanics is powerful. It reframes mechanics in terms of stationary action and gives a systematic way to derive equations of motion, especially for systems where coordinates, constraints, or paths matter more naturally than forces alone.

References

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